

# Representations of the $n$ -dimensional quantum torus

Ashish Gupta

## Abstract

The  $n$ -dimensional quantum torus  $\mathcal{O}_{\mathbf{q}}((F^\times)^n)$  is defined as the associative  $F$ -algebra generated by  $x_1, \dots, x_n$  together with their inverses satisfying the relations  $x_i x_j = q_{ij} x_j x_i$ , where  $\mathbf{q} = (q_{ij})$ . We show that the modules that are finitely generated over certain commutative sub-algebras  $\mathcal{B}$  are  $\mathcal{B}$ -torsion-free and have finite length. We determine the Gelfand-Kirillov dimensions of simple modules in the case when

$$\mathrm{K.dim}(\mathcal{O}_{\mathbf{q}}((F^\times)^n)) = n - 1,$$

where  $\mathrm{K.dim}$  stands for the Krull dimension. In this case if  $M$  is a simple  $\mathcal{O}_{\mathbf{q}}((F^\times)^n)$ -module then  $\mathcal{GK}\text{-dim}(M) = 1$  or

$$\mathcal{GK}\text{-dim}(M) \geq \mathcal{GK}\text{-dim}(\mathcal{O}_{\mathbf{q}}((F^\times)^n)) - \mathcal{GK}\text{-dim}(\mathcal{Z}(\mathcal{O}_{\mathbf{q}}((F^\times)^n))) - 1,$$

where  $\mathcal{Z}(C)$  stands for the center of an algebra  $C$ . We also show that there always exists a simple  $F * A$ -module satisfying the above inequality.

## 1 Introduction

The  $n$ -dimensional quantum torus  $\mathcal{O}_{\mathbf{q}}((F^\times)^n)$  is defined as the (associative)  $F$ -algebra which is generated by the variables  $x_1, x_2, \dots, x_n$  together with their inverses satisfying the relations

$$x_i x_j = q_{ij} x_j x_i \tag{1}$$

where  $1 \leq i, j \leq n$  and  $\mathbf{q} = (q_{ij})$ .

It plays an important role in non-commutative geometry and the theory of quantum groups. The case  $n = 2$  happens to be relatively well studied (see [10] and [8]). Here we consider the general case and in particular the structure and growth of modules.

It is well-known that the  $n$ -dimensional quantum torus has the structure of a twisted group algebra  $F * A$  of a free abelian group of rank  $n$  over the field  $F$ . The subgroups  $B \leq A$  so that the corresponding sub-algebra  $F * B$  commutative play an important role. For example we have the following result.

**Theorem 1.1** (Theorem A of [2]). *The Krull and global dimensions of a twisted group algebra  $F * A$  equals the supremum of the ranks of subgroups  $B \leq A$  so that  $F * B$  is commutative.*

Our first result describes the structure of the  $F * A$ -modules that are finitely generated over a sub-algebra  $F * B$  such that  $F * B$  is commutative.

**Theorem 4.1.** *Suppose that the quantum torus  $F * A$  has center  $F$ . Let  $M$  be a nonzero finitely generated  $F * A$ -module. Let  $C < A$  be a subgroup having a subgroup  $C_0$  of finite index such that  $F * C_0$  is commutative. If  $M$  is finitely generated as an  $F * C$ -module then,*

- (i)  $\mathcal{GK}\text{-dim}(M) = \text{rk}(C)$ ,
- (ii)  $M$  is  $F * C$ -torsion-free,
- (iii)  $M$  has finite length,
- (iv)  $M$  is cyclic.

### 1.1 Gelfand-Kirillov dimension of simple modules.

Let  $\mathcal{A}$  be an affine algebra of finite Gelfand–Kirillov dimension (GK dimension) say  $d$ . If  $M$  is a finitely generated  $\mathcal{A}$ -module then

$$0 \leq \mathcal{GK}\text{-dim}(M) \leq \mathcal{GK}\text{-dim}(\mathcal{A})$$

holds (e.g., [9, Proposition 5.1(d)]). However, not all values between 0 and  $\mathcal{GK}\text{-dim}(\mathcal{A})$  can be attained by the GK dimension of a finitely generated  $\mathcal{A}$ -module. For example, as is well known if  $\mathcal{A} = A_n(k)$  the  $n$ -th Weyl algebra over a field of characteristic zero, then the famous inequality of Bernstein says that

$$\mathcal{GK}\text{-dim}(M) \geq \frac{1}{2} \mathcal{GK}\text{-dim}(A_n(k)).$$

The question arises as to what values can be assumed by the GK dimensions of simple  $\mathcal{A}$ -modules. For example, if  $n \leq l \leq 2n - 1$  there exists a simple  $A_n(\mathbb{C})$ -module  $S_l$  with  $\mathcal{GK}\text{-dim}(S_l) = l$  (see [5]). The next question we take up concerns the GK dimension of simple modules over the quantum tori. In [12], it was shown that if  $\dim(F * A) = 1$  then each simple  $F * A$ -module  $N$  satisfies  $\mathcal{GK}\text{-dim}(N) = \text{rk}(A) - 1$ . Here  $\dim(F * A)$  stands for either the Krull or the global dimension of the quantum torus  $F * A$ .

Note that  $\dim(F * A)$  satisfies

$$1 \leq \dim(F * A) \leq \text{rk}(A).$$

If  $\dim(F * A) = \text{rk}(A)$  then by Theorem 1.1 the algebra  $F * A$  is a finite normalizing extension of  $F * A'$ , where the sub-algebra  $F * A'$  is commutative and so (e.g., [15, Exercise 15A.3]) a simple  $F * A$ -module is a finite direct sum of simple  $F * A'$ -modules. Using this and the fact that simple modules over commutative affine algebras are finite dimensional it can be easily deduced that in this case  $\mathcal{GK}\text{-dim}(S) = 0$  for each simple  $F * A$ -module  $S$ .

Here we consider the case  $\dim(F * A) = n - 1$ . We obtain the following results.

**Theorem 5.1.** *Let  $M$  be a simple- $F * A$ -module, where  $\dim(F * A) = \text{rk}(A) - 1$ . Let  $Z$  be the subgroup of  $A$  such that  $\mathcal{Z}(F * A) = F * Z$ . Then either*

$$\mathcal{GK}\text{-dim}(M) = 1, \text{ or}$$

$$\mathcal{GK}\text{-dim}(M) \geq \text{rk}(A) - \text{rk}(Z) - 1.$$

There always exists a simple  $F * A$ -module satisfying the inequality given in the last theorem.

**Theorem 5.3.** *Suppose that  $\dim(F * A) = \text{rk}(A) - 1$ . There exists a simple  $F * A$ -module  $M$  with*

$$\mathcal{GK}\text{-dim}(M) \geq \text{rk}(A) - \text{rk}(Z) - 1,$$

where  $Z$  is the unique subgroup of  $A$  so that

$$\mathcal{Z}(F * A) = F * Z.$$

Finally, we remark that it seems to be quite difficult to determine the possible values of GK-dimensions of simple modules over the quantum tori  $F * A$  such that

$$2 \leq \dim(F * A) \leq n - 2$$

It seems to depend on the defining parameters  $q_{ij}$

## 2 The twisted group algebra structure

Let  $F^* := F \setminus \{0\}$ . Let  $A$  denote a finitely generated free abelian group. We denote by  $\text{rk}(A)$  the (torsion-free) rank of  $A$ . An  $F$ -algebra  $\mathcal{A}$  is a *twisted group algebra*  $F * A$  of  $A$  over  $F$  if  $\mathcal{A}$  has a copy  $\overline{A} := \{\bar{a} : a \in A\}$  of  $A$  which is an  $F$ -basis and such that the multiplication in  $\mathcal{A}$  satisfies

$$\bar{a}_1 \bar{a}_2 = \tau(a_1, a_2) \overline{a_1 a_2} \quad \forall a_1, a_2 \in A, \quad (2)$$

where  $\tau : A \times A \rightarrow F^*$  is a function satisfying

$$\tau(a_1, a_2) \tau(a_1 a_2, a_3) = \tau(a_2, a_3) \tau(a_1, a_2 a_3) \quad \forall a_1, a_2, a_3 \in A.$$

For  $a_1, a_2 \in A$ , it easily follows from (2) that the group-theoretic commutator  $[\bar{a}_1, \bar{a}_2] \in F^*$ .

### 2.1 Commutator calculus

It follows from the basic properties of commutators (e.g., [14, Section 5.1.5]) that

$$[\bar{a}_1 \bar{a}_2, \bar{a}_3] = [\bar{a}_1, \bar{a}_3] [\bar{a}_2, \bar{a}_3], \quad (3)$$

$$[\bar{a}_1, \bar{a}_2 \bar{a}_3] = [\bar{a}_1, \bar{a}_2] [\bar{a}_1, \bar{a}_3], \quad (4)$$

$$[\bar{a}_1, \bar{a}_2^{-1}] = [\bar{a}_1, \bar{a}_2]^{-1}, \quad (5)$$

$$[\bar{a}_1^{-1}, \bar{a}_2] = [\bar{a}_1, \bar{a}_2]^{-1} \quad \forall a_1, a_2, a_3 \in A. \quad (6)$$

For a subset  $X$  of  $A$ , we define  $\overline{X} = \{\bar{x} : x \in X\}$ . If  $X, Y \subset A$ , we set

$$[\overline{X}, \overline{Y}] = \langle [\bar{x}, \bar{y}] : x \in X, y \in Y \rangle.$$

If  $\alpha \in F * A$ , we may express  $\alpha = \sum_{a \in A} \lambda_a \bar{a}$ , where  $\lambda_a \in F$  and  $\lambda_a = 0$  for almost all  $a \in A$ . We define the support of  $\alpha$  (in  $A$ ) as

$$\text{Supp}(\alpha) = \{a \in A \mid \lambda_a \neq 0\}.$$

Note that for a subgroup  $B$  of  $A$ , the sub-algebra generated by  $\overline{B} \subset F * A$  is a twisted group algebra  $F * B$ . It is described as

$$F * B = \{\beta \in F * A \mid \text{Supp}(\beta) \subset B\}$$

## 2.2 The center

It was shown in [12, Proposition 1.3] that an algebra  $F * A$  is simple if and only if it has center  $F$ . The following fact might be quite expected.

**Proposition 2.1.** *An algebra  $F * A$  has center exactly  $F$  if and only if for each subgroup  $A_1 < A$  with finite index  $F * A_1$  has center  $F$ .*

*Proof.* Suppose that  $F * A$  has center  $F$ . Let  $A_1 \leq A$  be a subgroup such that  $l := [A : A_1] < \infty$ . We claim that  $F * A_1$  also has center  $F$ . Using [12, Proposition 1.3], we may assume that  $\bar{a}_1$  is central in  $F * A_1$  for  $1 \neq a_1 \in A_1$ . For any  $a \in A$ , (3) and (4) yield:

$$[\bar{a}_1^l, \bar{a}] = [\bar{a}_1, \bar{a}]^l = [\bar{a}_1, \bar{a}^l] = 1,$$

where the last equality holds since  $a^l \in A_1$ . Since  $A$  is torsion-free by definition,  $1 \neq a_1^l$ . Thus  $\bar{a}_1^l$  is a non-scalar central element of  $F * A$ . The converse is clear.  $\square$

It may also be expected that the center of a twisted group algebra has the form  $F * B$  for a subgroup  $B \leq A$ .

**Proposition 2.2.** *The center of a twisted group algebra  $F * A$  has the form  $F * B$  for a suitable subgroup  $B$  in  $A$ .*

*Proof.* Let  $\mathcal{Z}$  be the center of the algebra  $F * A$  and  $\zeta \in \mathcal{Z}$ . Write  $\zeta = \sum_{i=1}^t \lambda_i z_i$ . Let  $A = \langle x_1, \dots, x_n \rangle$ . The condition  $\zeta \in \mathcal{Z}$  is equivalent to the  $n$  conditions  $[\bar{x}_i, \zeta] = 0$ , where  $1 \leq i \leq n$  and  $[a, b]$  stands for the lie commutator  $[a, b] = ab - ba$ . Now

$$\begin{aligned} [\bar{x}_i, \zeta] &= [\bar{x}_i, \sum_{j=1}^t \lambda_j a_j] \\ &= \sum_{j=1}^t \lambda_j [x_i, a_j]. \end{aligned}$$

From the defining relations we have  $[\bar{x}, \bar{y}] = \gamma(x, y) \overline{xy}$ , where  $\gamma(x, y) \in F$ . Therefore

$$[\bar{x}_i, \zeta] = \sum_{j=1}^t \lambda_j \gamma(\bar{x}_i, \bar{a}_j) \overline{x_i a_j}.$$

Hence  $[x_i, \zeta] = 0$  if and only if  $\gamma(\bar{x}_i, \bar{a}_j) = 0$ . But this means that  $[\bar{x}_i, \bar{a}_j] = 0$  and so  $\bar{a}_j \in \mathcal{Z}$ . Thus  $\mathcal{Z} = F * B$  where

$$B := \langle \cup \text{Supp}(\zeta) \mid \zeta \in \mathcal{Z} \rangle.$$

$\square$

## 2.3 Localization

It is well-known (see, for example, [13, Lemma 37.8]) that for subgroups  $B \leq A$  the subsets  $\mathcal{X}_B := F * B \setminus \{0\}$  are Ore subsets in the algebra  $F * A$  and the

latter has a (right) Ore localization with respect to  $\mathcal{X}_B$  which we denote as  $(F * A)\mathcal{X}_B^{-1}$ .

The localization has the structure of a crossed product of the group  $A/B$  over the quotient division ring  $\mathcal{D}_B$  of  $F * B$ . We refer the reader to [13] for details on crossed products. Recall that if  $M$  is an  $F * A$ -module the corresponding localization  $M(\mathcal{X})^{-1}$  is a module for  $(F * A)\mathcal{X}_B^{-1}$ . We refer the reader to [6] for a discussion of localization in a general non-commutative setting.

Let  $M$  be an  $F * A$ -module and  $C \leq A$ . As we noted above  $\mathcal{X} := F * C \setminus \{0\}$  is an Ore subset in  $F * A$ . The subset  $T_{\mathcal{X}}(M)$  of  $M$  defined by

$$T_{\mathcal{X}}(M) := \{x \in M \mid m.x = 0, x \in X\}$$

is a submodule of  $M$  and is known as the  $\mathcal{X}$ -torsion submodule of  $M$ . We will abuse notation somewhat and call it the  $F * C$ -torsion submodule of  $M$ .

### 3 The GK dimension of an $F * A$ -module

In this section we shall describe a dimension for finitely generated  $F * A$ -modules introduced in [3] that coincides with the GK dimension. More precisely, given a finitely generated  $F * A$ -module  $M$  it is shown in [3] that  $\mathcal{GK}\text{-dim}(M)$  is the supremum of the ranks of subgroups  $B \leq A$  so that  $M$  is not  $F * B$ -torsion. We shall use the notation

$$B \bowtie M$$

to denote the fact that  $M$  is not  $F * B$ -torsion. In [3] it was also shown that for each choice of (free) generators  $A : \langle x_1, x_2, \dots, x_n \rangle$  there is a subset  $\langle x_{i_1}, x_{i_2}, \dots, x_{i_k} \rangle$  of the generators so that

$$\langle x_{i_1}, x_{i_2}, \dots, x_{i_k} \rangle \bowtie M$$

and  $k = \mathcal{GK}\text{-dim}(M)$ . In this case we may localize  $M$  at the nonzero elements of  $F * \langle x_{i_1}, x_{i_2}, \dots, x_{i_k} \rangle$  and the nonzero module of fractions thus obtained is necessarily finite dimensional as a vector space over the quotient division ring of  $F * \langle x_{i_1}, x_{i_2}, \dots, x_{i_k} \rangle$ . In this situation the following holds.

**Lemma 3.1** (Lemma 4.1 of [7]). *Suppose that  $F * A$  has a finitely generated module  $M$  and  $A$  has a subgroup  $C$  with  $A/C$  torsion-free,  $\text{rk}(C) = \mathcal{GK}\text{-dim}(M)$ , and  $F * C$  commutative. Suppose moreover that  $M$  is not  $F * C$ -torsion. Then  $C$  has a virtual complement  $E$  in  $A$  such that  $F * E$  is commutative. Furthermore given  $\mathbb{Z}$ -bases  $\{x_1, \dots, x_r\}$  and  $\{x_1, \dots, x_r, x_{r+1}, \dots, x_n\}$  for  $C$  and  $A$  respectively there exist monomials  $\mu_j$ , where  $j = r + 1, \dots, n$ , in  $F * C$ , and an integer  $s > 0$  such that the monomials  $\mu_j \bar{x}_j^s$  commute in  $F * A$ .*

**Remark 3.2.** *In the case  $A/C$  is not torsion-free  $A$  has a subgroup  $A_1$  of finite index so that  $A_1/C$  is torsion-free. Moreover  $M$  is also a finitely generated  $F * A_1$ -module with the same GK dimension. We may thus apply the foregoing lemma which gives a virtual complement  $E_1$  of  $C$  in  $A_1$  so that  $F * E_1$  is commutative. But then  $E_1$  is also a virtual complement of  $C$  in  $A$ . Hence the first part of Lemma 3.1 holds true even when  $A/C$  is not torsion-free.*

In practice it is much more useful to work with the so-called critical modules.

**Definition 3.3.** An  $F * A$ -module is said to be *critical* if  $M$  is nonzero and every proper quotient  $M/N$  of  $M$  satisfies

$$\mathcal{GK}\text{-dim}(M/N) < \mathcal{GK}\text{-dim}(M), \quad 0 < N < M.$$

In [3] the following property was shown for the GK dimension of  $F * A$ -modules.

**Proposition 3.4** (Lemma 2.2 of [3]). *Let  $M$  be an  $F * A$ -module with a submodule  $N$ . Then*

$$\mathcal{GK}\text{-dim}(M) = \max(\mathcal{GK}\text{-dim}(N), \mathcal{GK}\text{-dim}(M/N))$$

The usefulness of the concept of a critical module owes itself to the following fact.

**Proposition 3.5** (Lemma 2.5 of [3]). *Each nonzero  $F * A$ -module contains a critical submodule.*

## 4 Finite length modules

We have the following result.

**Theorem 4.1.** *Let  $M$  be a nonzero finitely generated  $F * A$ -module where  $F * A$  has center  $F$ . Let  $C < A$  be a subgroup having a subgroup  $C_0$  of finite index such that  $F * C_0$  is commutative. If  $M$  is finitely generated as an  $F * C$ -module then,*

- (i)  $\mathcal{GK}\text{-dim}(M) = \text{rk}(C)$ ,
- (ii)  $M$  is  $F * C$ -torsion-free,
- (iii)  $M$  has finite length.

*Proof.* (i) We shall denote the module  $M$  regarded as  $F * C$ -module as  $M_C$ . By hypothesis  $M_C$  is finitely generated and so  $M_{C_0}$  is also finitely generated where  $M_{C_0}$  denotes  $M_C$  viewed as  $F * C_0$ -module. By [3, Lemma 2.7],

$$\mathcal{GK}\text{-dim}(M) = \mathcal{GK}\text{-dim}(M_C) = \mathcal{GK}\text{-dim}(M_{C_0}).$$

If  $\mathcal{GK}\text{-dim}(M) < \text{rk}(C)$  we may pick a subgroup  $E_0 < C_0$  with  $\text{rk}(E_0) < \text{rk}(C_0)$  such that  $E_0 \bowtie M_{C_0}$  and  $\mathcal{GK}\text{-dim}(M) = \text{rk}(E_0)$ .

By Lemma 3.1 and Remark 3.2,  $E_0$  has a virtual complement  $E_1$  in  $A$  such that  $F * E_1$  is commutative. Since  $\text{rk}(E_1) + \text{rk}(C_0)$  exceeds  $\text{rk}(A)$ , therefore  $E_1 \cap C_0 > \langle 1 \rangle$ . Moreover as  $E_1 E_0$  has finite index in  $A$ , hence  $E_1 C_0$  has finite index in  $A$ . But  $\overline{E_1 \cap C_0}$  is central in  $F * (E_1 C_0)$  and hence by Proposition 2.1,  $F * A$  has center larger than  $F$ . This is contrary to the hypothesis in the theorem. So

$$\mathcal{GK}\text{-dim}(M) \geq \text{rk}(C).$$

But the GK-dimension of an algebra  $\mathcal{A}$  bounds the GK-dimensions of its modules ([9][Proposition 5.1(d)]) and so

$$\mathcal{GK}\text{-dim}(M) \leq \mathcal{GK}\text{-dim}(F * C) = \text{rk}(C).$$

(ii) Suppose that the  $F * C$ -torsion submodule  $T$  of  $M$  is nonzero. We recall that  $T$  is an  $F * A$ -submodule of  $M$ . Applying part (i) of the theorem just established to  $T$  we obtain  $\mathcal{GK}\text{-dim}(T) = \text{rk}(C)$ .

From Section 3 we know that in this case  $C \bowtie T$ . However, this is contrary to the definition of  $T$  as the  $F * C$ -torsion submodule of  $M$ . Hence  $T = 0$  and  $M$  is  $F * C$ -torsion-free.

(iii) We first note that by part (i) and Proposition 3.4 each nonzero subfactor of  $M$  has the same GK dimension as  $M$ . It now follows from [12, Lemma 5.6] and [12, Section 5.9] that every descending chain in  $M$  is eventually constant.  $\square$

## 5 The GK dimensions of simple modules

In this section we aim to determine the GK dimensions of simple  $F * A$ -modules. This may depend to a large extent on the defining parameters  $q_{ij}$ . In [12] it was shown that when  $\dim(F * A) = 1$ , then each simple  $F * A$ -module has GK dimension equal to  $\text{rk}(A) - 1$ .

In general the GK dimensions of simple  $F * A$ -modules depends on the defining co-cycle. But in the case  $\dim(F * A) = \text{rk}(A) - 1$ , we have the following result.

**Theorem 5.1.** *Let  $M$  be a simple- $F * A$ -module, where  $\dim(F * A) = \text{rk}(A) - 1$ . Let  $Z$  be the subgroup of  $A$  such that  $\mathcal{Z}(F * A) = F * Z$ . Then either  $\mathcal{GK}\text{-dim}(M) = 1$  or*

$$\mathcal{GK}\text{-dim}(M) \geq \text{rk}(A) - \text{rk}(Z) - 1.$$

**Remark 5.2.** *We observe that  $\text{rk}(Z) \leq \text{rk}(A) - 2$ . Otherwise  $A$  contains a subgroup  $A'$  of finite index such that  $F * A'$  is commutative but this would contradict Theorem 1.1.*

*Proof.* Since  $\dim(F * A) = \text{rk}(A) - 1$ , by Theorem 1.1  $A$  contains a subgroup  $C$  with co-rank one such that  $F * C$  is commutative. However note that  $A/C$  need not be infinite cyclic. In the case it is

$$F * A = (F * C)[X_n^{\pm 1}, \sigma], \tag{7}$$

that is,  $F * A$  is a skew-Laurent extension of  $F * C$ . Here  $\sigma$  is the automorphism of  $F * C$  defined by  $\gamma \mapsto X_n \gamma X_n^{-1}$ , where  $X_n$  denotes the image of a generator of  $A$  modulo  $C$ .

In the general case let  $C$  be embedded in a subgroup  $A'$  so  $A'/C$  is infinite cyclic. Then the remarks made above apply to  $F * A'$ . Now  $F * A$  is a finite normalizing extension of  $F * A'$  and so (e.g., [15, Exercise 15A.3]) a simple  $F * A$ -module is a finite direct sum of simple  $F * A'$ -modules. As the GK dimension of a direct sum is the maximum of the GK dimensions of its summands ([9, Proposition 5.1]), it suffices to prove the theorem for  $F * A'$ . We may thus assume that  $A/C$  is infinite cyclic.

Now let  $S$  be a simple  $F * A$ -module. Let  $L$  be a finitely generated critical  $F * C$ -submodule of  $S$ . Consider the  $F * A$  submodule  $L_1 := L(F * A)$ . Since  $S$  is simple  $L_1 = S$ .

Now  $L_1 = \sum_{i \in \mathbb{Z}} L X^i$ . There is a surjective map  $\theta$  from the induced module

$$\oplus_{i \in \mathbb{Z}} L X^i \cong L \otimes_{F * B} F * A$$

to  $L_1$ . If this map is an isomorphism then  $L$  is also simple and

$$\mathcal{GK}\text{-dim}(L_1) = \mathcal{GK}\text{-dim}(L) + 1$$

by [3, Lemma 2.3]. Since  $F * C$  is commutative affine algebra  $L$  is finite dimensional (over  $F$ ) by the Nullstellensatz (see [11, Section ]). As the GK dimension of a finite dimensional module is zero, it follows that  $\mathcal{GK}\text{-dim}(L_1) = 1$ .

Now suppose that  $\ker \theta \neq 0$ . Then  $\mathcal{GK}\text{-dim}(L) = \mathcal{GK}\text{-dim}(L_1)$  by [3, Lemma 2.4]. Let  $l := \mathcal{GK}\text{-dim}(L_1)$  and  $C = \langle y_1, y_2, \dots, y_{n-1} \rangle$ . Since  $A/C$  is infinite cyclic we may write  $A = \langle y_1, y_2, \dots, y_{n-1}, y_n \rangle$ .

By the criterion for the GK dimension of an  $F * A$ -module (Section 3) we can pick  $l$  generators, say  $y_1, y_2, \dots, y_l$  such that

$$\langle y_1, y_2, \dots, y_l \rangle \bowtie L.$$

But then

$$\langle y_1, y_2, \dots, y_l \rangle \bowtie L_1.$$

By Lemma 3.1, there are monomials  $\mu_1, \mu_2, \dots, \mu_{n-l}$  in  $\bar{y}_1, \dots, \bar{y}_l$  so that

$$\mu_1 \bar{y}_{l+1}^s, \mu_2 \bar{y}_{l+2}^s, \dots, \mu_{n-l} \bar{y}_n^s$$

is a system of  $n-l$  independent commuting monomials. It is clear that the first  $n-l-1$  monomials centralize  $F * C$  since  $F * C$  is commutative.

Next we observe that if  $1 \leq t \leq n-l-1$  then

$$1 = [\mu_t \bar{y}_{l+t}^s, \mu_{n-l} \bar{y}_n^s] = [\mu_t \bar{y}_{l+t}^s, \bar{y}_n^s] = [(\mu_t \bar{y}_{l+t}^s)^s, \bar{y}_n] = [\mu_t^s \bar{y}_{l+t}^{s^2}, \bar{y}_n]$$

noting Section 2.1. Hence the system  $\{\mu_t^s \bar{y}_{l+t}^{s^2}\}_{t=1}^{n-l-1}$  of independent monomials centralizes the algebra  $F * A$ .

It follows that

$$\text{rk}(Z) \geq n-l-1$$

which gives the desired inequality.  $\square$

**Theorem 5.3.** *Suppose that  $\dim(F * A) = \text{rk}(A) - 1$ . There exists a simple  $F * A$ -module  $M$  with*

$$\mathcal{GK}\text{-dim}(M) \geq \text{rk}(A) - \text{rk}(Z) - 1,$$

where  $Z$  is the unique subgroup of  $A$  so that

$$Z(F * A) = F * Z.$$

*Proof.* Set  $\mathcal{S} = F * C \setminus \{0\}$ . Then using equation (7), the localization  $(F * A)\mathcal{S}^{-1}$  is a skew-Laurent extension

$$R := (F * A)\mathcal{S}^{-1} = \mathcal{D}_C[X^{\pm 1}, \sigma]$$

where  $\mathcal{D}_C$  stands for the quotient division ring of  $F * C$  and  $\sigma$  is the automorphism of  $\mathcal{D}_C$  defined by  $\sigma(f) = X_n f X_n^{-1}$  for  $f \in \mathcal{D}_C$ . Let  $r$  be an irreducible element in the non-commutative PLID  $R$  such that  $(F * A) \cap rR$  contains a unitary element of  $F * A$ , that is, an element  $\alpha$  such that in the unique expression



$\alpha = \sum_{i=l}^h \gamma_i X_n^i$  the coefficients of the lowest and the highest powers of  $x_n$  are units in  $F * C$ .

It is not difficult to find examples of such elements  $r$ . For example,  $u_1 X_n + u_2$ , where  $u_i$  is a unit in  $F * C$  is a linear polynomial in  $\mathcal{D}_C[X^{\pm 1}, \sigma]$  and so is irreducible. Let

$$M(r) := (F * A) / J(r),$$

where  $J(r) := (F * A) \cap rR$ . Note that  $rR$  is a maximal right ideal in  $R$ . By [4, Lemma 3.4(1)], we know that  $M(r)$  is a simple  $F * A$ -module if and only if

$$\text{Hom}_{F * A}(M(r), N) = 0$$

for all  $F * C$ -torsion simple  $F * A$ -modules  $N$ . Suppose this last condition holds true so that  $M(r)$  is a  $F * C$ -torsion-free simple  $F * A$ -module. Moreover as it is  $F * C$ -torsion-free

$$\mathcal{GK}\text{-dim}(M(r)) \geq \text{rk}(C) = n - 1.$$

Otherwise by [1, Proposition 2.1],  $M(r)$  and so  $N$  is finitely generated as  $F * C$ -module. Now a reasoning similar to the one given in the proof of the last theorem with  $N$  in place of  $L$  shows that

$$\mathcal{GK}\text{-dim}(N) \geq \text{rk}(A) - \text{rk}(Z) - 1.$$

□

## References

- [1] Artamonov, V. A. General quantum polynomials: irreducible modules and Morita equivalence. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* 63 (1999), no. 5, 3–36
- [2] Brookes, C. J. B., Crossed products and finitely presented groups. *J. Group Theory* 3 (2000), no. 4, 433–444.
- [3] C. J. B. Brookes, J. R. J. Groves, Modules over crossed products of a division ring with an abelian group I, *J. Algebra*, 229, 2000, pp. 25–54.
- [4] Bavula, V., van Oystaeyen, F., Simple holonomic modules over the second Weyl algebra A2. *Adv. Math.* 150 (2000), no. 1, 80–116.
- [5] Coutinho, S. C., A primer of algebraic D-modules. London Mathematical Society Student Texts, 33. Cambridge University Press, Cambridge, 1995. xii+207 pp.
- [6] Goodearl, K. R., Warfield, R. B., Jr. An introduction to noncommutative Noetherian rings. London Mathematical Society Student Texts, 16. Cambridge University Press, Cambridge, 1989.
- [7] Gupta, A., Modules over quantum Laurent polynomials., *J. Aust. Math. Soc.* 91 (2011), no. 3, 323–341.
- [8] V. A. Jategaonkar, ‘A multiplicative analogue of the Weyl algebra’, *Comm. Algebra* 12 (1984), 1669–1688.

- [9] G. R. Krause, T. H. Lenagan, Growth of Algebras and Gelfand-Kirillov Dimension, American Mathematical Soc. 2000.
- [10] M. Lorenz, *Group Rings and Division Rings, Methods in Ring Theory*, 265-280 (D. Reidel Publ. Co., Dordrecht, 1984).
- [11] McConnell, J. C., Robson, J. C., Noncommutative Noetherian rings. With the cooperation of L. W. Small. Revised edition. Graduate Studies in Mathematics, 30. American Mathematical Society, Providence, RI, 2001
- [12] J. C. McConnell and J. J. Pettit, 'Crossed products and multiplicative analogues of Weyl Algebras', *J. London Math. Soc.* **38** (1988), 47–55.
- [13] Passman, D. S., Infinite crossed products, Pure and Applied Mathematics, 135. Academic Press, Inc., Boston, MA, 1989. xii+468 pp.
- [14] Derek J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1996.
- [15] Rowen, L. H., Graduate algebra: noncommutative view. Graduate Studies in Mathematics, 91. American Mathematical Society, Providence, RI, 2008.